

An Exceptional Representation of $Sp(4, \mathbb{F}_q)$

Tanmay Deshpande

1 Introduction

Let p be an odd prime, and let q be a power of p . Let $G = Sp(4)$. B.Srinivasan (in [5]) discovered an irreducible representation (denoted by θ_{10}) of $Sp(4, \mathbb{F}_q)$ with the following remarkable combination of properties, namely it is cuspidal(Defn. 4.1), unipotent(Defn. 5.8) as well as degenerate, i.e. it does not admit a Whittaker model(Defn. 4.2). The groups $SL_n(\mathbb{F}_q)$ and $GL_n(\mathbb{F}_q)$ do not have any unipotent cuspidal representations and neither do they have any degenerate cuspidal representations. Hence the existence of such a representation for $Sp(4, \mathbb{F}_q)$ is somewhat surprising.

We will describe a folklore construction of θ_{10} , which is different from [5]. It is based on the Weil representation of $Sp(8, \mathbb{F}_q)$ and Howe duality. This article was a part of my master's thesis during my graduate studies at the University of Chicago. My advisor, V.Drinfeld, suggested that I publish this article in the e-print archive, since there were apparently no references for this construction of θ_{10} .

2 Construction

Let V be a four-dimensional symplectic vector space over \mathbb{F}_q with a symplectic form $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_V$. Let $E = \mathbb{F}_{q^2}$ considered as a two-dimension vector space over \mathbb{F}_q with a non-degenerate symmetric bilinear form corresponding to the norm, namely given by $\langle x, y \rangle_E = \frac{1}{2}(xy^q + x^qy)$. This is an anisotropic bilinear form. Moreover, any two-dimensional vector space over \mathbb{F}_q with an anisotropic quadratic form is isomorphic to E . We see that the eight-dimensional space $V \otimes E$ inherits a natural symplectic form. Moreover, we have a natural map from $Sp(V) \times O(E)$ to $Sp(V \otimes E)$. Note that the group $SO(E)$ is just the cyclic group of order $q+1$, consisting of the norm 1 elements of $\mathbb{F}_{q^2}^*$.

Let ψ be a non-trivial character $\psi : \mathbb{F}_q \rightarrow \overline{\mathbb{Q}}_l^*$, where l is a prime different from p . Let us now consider the corresponding Weil representation, W of $Sp(V \otimes E)$. Thus we get an action of $Sp(V) \times O(E)$ on W . Let $L, L' \subset V$ be complementary Lagrangian subspaces. Then we can identify W with the space of $\overline{\mathbb{Q}}_l$ -valued functions on $L' \otimes E$. Then for $t \in O(E) \hookrightarrow Sp(V \otimes E)$, $v \in L' \otimes E$ and $f : L' \otimes E \rightarrow \overline{\mathbb{Q}}_l$, we have $(t \cdot f)(v) = f((1 \otimes t^{-1})(v))$ (See 4.11). For a character θ of $SO(E)$, let W_θ denote the θ -isotypic part of W , i.e. $W_\theta = \{f | f((1 \otimes t^{-1})(v)) = \theta(t)f(v) \text{ for all } t \in SO(E), v \in L' \otimes E\}$. Then W_θ is a representation of $Sp(V)$. We see that

$$\dim(W_\theta) = \begin{cases} \frac{q^4-1}{q+1} + 1 = q(q^2 - q + 1) & \text{if } \theta = 1 \\ \frac{q^4-1}{q+1} = (q-1)(q^2 + 1) & \text{else.} \end{cases} \quad (1)$$

Note that we have the element 'conjugation', $(\sigma : \mathbb{F}_{q^2} \rightarrow \mathbb{F}_{q^2}) \in O(E)$. Let $\Lambda : W \rightarrow W$ denote the action of this element. Then Λ commutes with the action of $Sp(V)$ and takes W_θ isomorphically to $W_{\theta^{-1}}$. Let ν be the quadratic character of $SO(E)$. Then for $\theta = 1$ or ν , we have $\Lambda : W_\theta \rightarrow W_\theta$, and $\Lambda^2 = 1$. Let W_θ^\pm be the (± 1) -eigenspace of $\Lambda|_{W_\theta}$. Let us now describe the dimensions of these four representations.

Lemma 2.1. $\dim(W_\nu^\pm) = \frac{1}{2}(q-1)(q^2+1)$, $\dim(W_1^+) = \frac{1}{2}q(q^2+1)$ and $\dim(W_1^-) = \frac{1}{2}q(q-1)^2$.

Proof. From the way $O(E)$ acts on W , we see that $(\Lambda \cdot f)(v) = f((1 \otimes \sigma)(v))$, where $f \in W, v \in L' \otimes E$. Hence the trace of the operator $\Lambda =$ number of fixed points of $1 \otimes \sigma : L' \otimes E \rightarrow L' \otimes E$. Now σ is diagonalizable, with eigenvalues $\{1, -1\}$. Hence number of fixed points of $1 \otimes \sigma =$ number of elements in the 1-eigenspace of

$1 \otimes \sigma = q^2$. Hence the trace of $\Lambda = q^2$. Similarly, W_1 can be identified with the space of $\overline{\mathbb{Q}_l}$ -valued functions on $(L' \otimes E)/SO(E)$, and as before the trace of $\Lambda|_{W_1} = \text{number of fixed points of } 1 \otimes \sigma : (L' \otimes E)/SO(E) \rightarrow (L' \otimes E)/SO(E) = |S/SO(E)|$, where the set $S = \{v \in L' \otimes E | (1 \otimes \sigma)(v) = (1 \otimes \lambda)(v) \text{ for some } \lambda \in SO(E)\}$ i.e. $S/SO(E)$ is precisely the set of self-conjugate orbits. Let (v'_1, v'_2) be a basis of L' . Then any vector $v \in L' \otimes E$ can be uniquely written as $v = v'_1 \otimes e_1 + v'_2 \otimes e_2$, where $e_1, e_2 \in E$ are uniquely determined by v . Then such a $v \in S \iff v'_1 \otimes e_1^q + v'_2 \otimes e_2^q = v'_1 \otimes \lambda e_1 + v'_2 \otimes \lambda e_2$ for some $\lambda \in SO(E) \iff e_1^{q-1} = e_2^{q-1}$ or one of $e_1, e_2 = 0 \iff e_1, e_2$ linearly dependent $\iff v$ is decomposable, i.e. $v = v' \otimes e$ for some $v' \in L', e \in E$. Hence S is precisely the set of decomposable vectors. Now the number of non-zero decomposable vectors in $L' \otimes E = \frac{(q^2-1)^2}{q-1}$ and hence $|S \setminus \{0\}|/SO(E) = q^2 - 1$. So we see that $|S/SO(E)| = q^2$, and hence trace of $\Lambda|_{W_1} = q^2$ as well. Hence from the way Λ acts on W , we see that trace of $\Lambda|_{W_\nu} = 0$. The lemma now follows from (1), since the trace of $\Lambda|_{W_\theta} = \dim(W_\theta^+) - \dim(W_\theta^-)$ for $\theta = 1$ or ν . \square

Proposition 2.2. *The representations $W_\theta \cong W_{\theta^{-1}}$ for $\theta \neq 1, \nu$, W_1^\pm and W_ν^\pm are all irreducible and distinct.*

Proof. We see that as an $Sp(V)$ representation we have

$$W = 2 \cdot \left(\bigoplus_{\langle \theta \rangle, \theta \neq 1, \nu} W_\theta \right) \oplus W_1^+ \oplus W_1^- \oplus W_\nu^+ \oplus W_\nu^-. \quad (2)$$

Hence we must have $\langle W, W \rangle_{Sp(V)} \geq 4(q-1)/2 + 4 = 2q + 2$, since all the summands above are non-zero. On the other hand, we now show that $\dim(\text{End}_{Sp(V)}(W)) = 2q + 2$. Let \mathcal{A} be the group algebra of the Heisenberg group. Let \mathcal{A}_ψ denote the quotient of \mathcal{A} corresponding to the central character ψ . Since W is the space of the irreducible representation of the Heisenberg group with central character ψ , we get a canonical isomorphism $\mathcal{A}_\psi \xrightarrow{\sim} \text{End}_{\mathbb{C}}(W)$ that is $Sp(V \otimes E)$ -equivariant. Now as a representation of $Sp(V \otimes E)$, \mathcal{A}_ψ identifies with the space of $\overline{\mathbb{Q}_l}$ -valued functions on $V \otimes E$. Now $\text{End}_{Sp(V)}(W) = \text{End}_{\mathbb{C}}(W)^{Sp(V)}$. Hence we see that $\dim(\text{End}_{Sp(V)}(W)) = \dim((\text{End}_{\mathbb{C}}(W)^{Sp(V)}) = \dim(\mathcal{A}_\psi^{Sp(V)}) = \text{number of } Sp(V)\text{-orbits in } V \otimes E$. Let (e_1, e_2) be a basis of E . Then as before, an elements of $V \otimes E$ can be uniquely written in the form $v_1 \otimes e_1 + v_2 \otimes e_2$ with $v_1, v_2 \in V$. Then, we have the orbit $\{0\}$. The set of orbits of non-zero decomposable vectors can be identified with $\mathbb{P}(E)$, and finally the set of orbits of indecomposable elements can be identified with \mathbb{F}_q via the correspondence $Sp(V) \cdot (v_1 \otimes e_1 + v_2 \otimes e_2) \leftrightarrow \langle v_1, v_2 \rangle_V$. So we see that the number of orbits is exactly $2q + 2$ i.e. $\langle W, W \rangle_{Sp(V)} = 2q + 2$. Hence we conclude that all the summands in the decomposition above must be irreducible and distinct. \square

In particular, W_1^- is an irreducible $d = \frac{1}{2}q(q-1)^2$ -dimensional representation of $Sp(V)$. Prop.4.12 gives another proof of the irreducibility of this representation. For historical reasons, let us denote its character by θ_{10} . We will prove that this representation is cuspidal, degenerate and unipotent. Let us first study the space W_1^- . Let S' be the set of indecomposable vectors in $L' \otimes E$. As we have seen in the proof of 2.1, $S'/SO(E) \subset (L' \otimes E)/SO(E)$ is precisely the set of $SO(E)$ -orbits that are not self-conjugate. Let $\mathcal{O}_1, \mathcal{O}'_1, \mathcal{O}_2, \mathcal{O}'_2, \dots, \mathcal{O}_d, \mathcal{O}'_d$ be all such orbits, where $\mathcal{O}'_i = (1 \otimes \sigma)\mathcal{O}_i$. Then it is clear that the functions $\delta_i = \delta_{\mathcal{O}_i} - \delta_{\mathcal{O}'_i}$ form a basis of W_1^- , where for $X \subset L' \otimes E$, δ_X denotes the function that takes the value 1 on X and 0 elsewhere.

3 Parabolic Subgroups of $Sp(V)$

The Weyl group of $Sp(4)$ is isomorphic to the dihedral group D_8 . Let $0 \subset L_1 \subset L(=L^\perp) \subset L_1^\perp \subset V$ be a complete flag in V and let L' be a complementary Lagrangian subspace to the Lagrangian subspace $L \subset V$. Now the stabilizer B_0 of this complete flag is a Borel subgroup. Let U_0 be its unipotent radical. Then U_0 is a maximal unipotent subgroup and its order is q^4 . Let (v_1, v_2) be a basis of L such that $v_1 \in L_1$, and let (v_3, v_4) be a basis of L' such that the matrix of \langle, \rangle_V with respect to the basis (v_1, v_2, v_3, v_4) is $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$. Thus we have identified $Sp(V)$ with $Sp(4, \mathbb{F}_q)$. We see $v_1 \in L_1, v_2 \in L, v_3 \in L_1^\perp, v_4 \in V$. Then with respect

to this basis, U_0 is the group of matrices of the type $\begin{pmatrix} 1 & -\alpha & \beta & \mu \\ 0 & 1 & \lambda & \lambda\alpha+\beta \\ 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}$ where $(\lambda, \alpha, \mu, \beta) \in \mathbb{F}_q^4$. Let T_0 be the torus of diagonal matrices $\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b^{-1} & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix}$ in $Sp(V)$ with respect to this basis. Then we can identify¹ T_0 with $\mathbb{G}_m \times \mathbb{G}_m$, and hence we can identify $Hom(T_0, \mathbb{G}_m)$ with \mathbb{Z}^2 . With this identification, the roots of $Sp(4)$ are $\{\pm(1, -1), \pm(0, 2), \pm(1, 1), \pm(2, 0)\}$, and the choice of positive roots (implicit in this notation) is forced by our choice of the Borel subgroup. The simple roots are $r_1 = (1, -1)$ and $r_2 = (0, 2)$. The other positive roots are $r_3 = r_1 + r_2$ and $r_4 = 2r_1 + r_2$. The Weyl group $W(T_0)$ is generated by the two simple reflections $s_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ corresponding to the simple roots r_1, r_2 respectively. Let us now describe the various root subgroups.

$$\begin{aligned} \underline{U}_{r_1} &= \left\{ \begin{pmatrix} 1 & -\alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \underline{U}_{-r_1} = \underline{U}_{r_1}^T. \\ \underline{U}_{r_2} &= \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \underline{U}_{-r_2} = \underline{U}_{r_2}^T. \\ \underline{U}_{r_3} &= \left\{ \begin{pmatrix} 1 & 0 & \beta & 0 \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \underline{U}_{-r_3} = \underline{U}_{r_3}^T. \\ \underline{U}_{r_4} &= \left\{ \begin{pmatrix} 1 & 0 & 0 & \mu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \underline{U}_{-r_4} = \underline{U}_{r_4}^T. \end{aligned}$$

Let P_1 be the Siegel parabolic subgroup corresponding to the Lagrangian subspace L , i.e. the subgroup of elements of $Sp(V)$ that leave the sub-flag $0 < L < V$ invariant. This is the parabolic subgroup $B_0 \cup B_0 s_1 B_0$. The unipotent radical U_1 of P_1 consists of those elements of $Sp(V)$ that act as identity on L . We have $U_1 = U_{r_2} U_{r_3} U_{r_4}$. It consists of the matrices $\begin{pmatrix} 1 & 0 & \beta & \mu \\ 0 & 1 & \lambda & \beta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Lemma 3.1. U_1 can be naturally identified with the additive group of symmetric bilinear forms on L' .

Proof. For $g \in Sp(V)$, we have $\ker(g - 1)^\perp = \text{Im}(g - 1)$. Suppose $g \in U_1$. Then $L \subset \ker(g - 1)$ and $\text{Im}(g - 1) \subset L$. Thus $g - 1$ induces a map $A_g : L' \cong V/L \rightarrow L$. On the other hand, a bilinear form A on $L' = V/L$ given by a map $A : V/L \rightarrow L$, induces a map $A' : V \rightarrow V$. Then we have that $\langle (1 + A')v, (1 + A')w \rangle = \langle v, w \rangle \iff \langle v, A'w \rangle + \langle A'v, w \rangle = 0$. Hence we see that $1 + A' \in Sp(V) \iff A$ is a symmetric bilinear form on L' . Moreover, if A_1, A_2 are two symmetric bilinear forms on L' , then $(1 + A'_1)(1 + A'_2) = 1 + (A_1 + A_2)'$. Hence we have identified the group U_1 , with the group of symmetric bilinear forms on L' . \square

So we see that in fact, U_1 is a 3-dimensional vector space over \mathbb{F}_q . For $g \in U_1$, let $\langle \cdot, \cdot \rangle_g$ denote the corresponding bilinear form on L' .

Let P_2 be the stabilizer of the flag $0 < L_1 < L_1^\perp < V$. This is the parabolic subgroup $B_0 \cup B_0 s_2 B_0$. Let U_2 be its unipotent radical. Then $U_2 = \{g \in Sp(V) | (g - 1)L_1 = 0 \text{ and } (g - 1)L_1^\perp \subset L_1\}$. We have $|U_2| = q^3$.

We have $U_2 = U_{r_1} U_{r_3} U_{r_4}$. We see that U_2 consists of the matrices $\begin{pmatrix} 1 & -\alpha & \beta & \mu \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Now B_0, P_1 and P_2 are all the proper parabolic subgroups containing B_0 .

Let $U'_0 = U_1 \cap U_2$. Then U'_0 is the commutator subgroup of U_0 . For $g \in U_1$, we see that $g \in U_2 \iff \langle L_1^\perp, (g - 1)L_1^\perp \rangle = 0 \iff \langle L_1^\perp/L, L_1^\perp/L \rangle_g = 0$. Hence U'_0 can be identified with the group of all symmetric bilinear forms on L' such that $L_1^\perp/L \subset L'$ is an isotropic subspace with respect to that form. We have

$|U'_0| = q^2$. We have $U'_0 = U_{r_3} U_{r_4}$. It consists of the matrices $\begin{pmatrix} 1 & 0 & \beta & \mu \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Let $U''_0 \subset U'_0$ be the group of all

¹Here we use the convention that for certain nice $X \subset Sp(V)$, \underline{X} denotes the obvious F -stable subvariety of $Sp(4)$ such that $\underline{X}^F = X$.

symmetric bilinear forms that contain L_1^\perp/L in their kernels. Then in fact U_0'' is the center of U_0 . We have $U_0'' = U_{r_4} \cong \mathbb{F}_q$. It consists of the matrices $\begin{pmatrix} 1 & 0 & 0 & \mu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

4 Cuspidality and Degeneracy

Let us first recall the definitions of cuspidality and degeneracy for finite groups of Lie type.

Definition 4.1. Let G be a connected reductive group over $\overline{\mathbb{F}}_q$ that has an \mathbb{F}_q -structure given by a geometric Frobenius endomorphism F . Let ρ be a representation of G^F . We say that ρ is cuspidal, if for any proper F -stable parabolic subgroup P of G with unipotent radical U , $\rho|_{U^F}$ does not contain the trivial representation of U^F .

Let us now recall the definition of a Whittaker model. For simplicity, let us assume that the group G is split over \mathbb{F}_q . Let U_0 be a maximal F -stable unipotent subgroup of G contained in an F -stable Borel subgroup B_0 . Let $B_0 = T_0 U_0$, where T_0 is F -stable. The group $U_0/[U_0, U_0]$ is isomorphic to a sum of copies of the groups \mathbb{G}_a , one copy for each simple root of G with respect to the pair (T_0, B_0) . G being split implies that these copies of \mathbb{G}_a are F -stable. Let $\xi : U_0^F/[U_0, U_0]^F \rightarrow \overline{\mathbb{Q}}_l^*$ be a non-degenerate character, i.e. the restriction of ξ to each copy of \mathbb{G}_a^F is non-trivial. We can consider ξ as a character of U_0^F . We call such characters ξ of U_0^F as non-degenerate characters. Then the representation $\text{Ind}_{U_0^F}^{G^F}(\xi)$ is multiplicity free. (See [6], p. 258-262.)

Definition 4.2. Let ρ be an irreducible representation of G^F . We say that ρ admits a Whittaker model, if $\rho|_{U_0^F}$ contains a one-dimensional non-degenerate character ξ of U_0^F . We say that ρ is degenerate if it does not admit a Whittaker model.

Remark 4.3. By Frobenius reciprocity and the fact that $\text{Ind}_{U_0^F}^{G^F}(\xi)$ is multiplicity free, we see that if ρ_{U_0} contains such a non-degenerate ξ , then we must have $\langle \xi, \rho|_{U_0^F} \rangle_{U_0^F} = 1$.

Remark 4.4. If ρ is such that $\rho|_{U_0^F}$ does not contain any one-dimensional characters, then ρ must be degenerate.

Let us now return to the group $Sp(V)$. We now study how U_1 acts on the Weil representation W . Now $L \otimes E$ and $L' \otimes E$ are complementary Lagrangian subspaces of the symplectic vector space $V \otimes E$. So, as before, we can identify the space W with $\{f|f : L' \otimes E \rightarrow \overline{\mathbb{Q}}_l\}$. Let \mathcal{P} be the Siegel parabolic subgroup of $Sp(V \otimes E)$ corresponding to $L \otimes E$ i.e. the stabilizer of this Lagrangian subspace. Let \mathcal{U} be its unipotent radical. Then exactly as before, we may identify \mathcal{U} with the additive group of symmetric bilinear forms on $L' \otimes E$. We have the inclusion $Sp(V) \hookrightarrow Sp(V \otimes E)$ given by $g \mapsto g \otimes 1$. This induces the inclusions $P_1 \hookrightarrow \mathcal{P}$ and $U_1 \hookrightarrow \mathcal{U}$. In terms of bilinear forms, we have $\langle \cdot, \cdot \rangle_{g \otimes 1} = \langle \cdot, \cdot \rangle_g \otimes \langle \cdot, \cdot \rangle_E$ for $g \in U_1$. We now recall how \mathcal{U} acts on W .

Proposition 4.5 (See [2], 2.8.). *Let $g \in \mathcal{U}$ and $f : L' \otimes E \rightarrow \overline{\mathbb{Q}}_l$. Then*

$$(g \cdot f)(v) = \psi \left(\frac{\langle v, v \rangle_g}{2} \right) f(v). \quad (3)$$

The cuspidality and degeneracy of W_1^- follow immediately from the following:

Lemma 4.6. *The restriction $W_1^-|_{U_0'}$ does not contain the trivial representation of U_0' , i.e. W_1^- has no non-zero U_0' -fixed vector. In fact, W_1^- does not even have any non-zero U_0'' -fixed vectors.*

Proof. Let $f \in W$ be a fixed vector for the action U_0'' . Suppose $v \in L' \otimes E$ is such that $f(v) \neq 0$. Then from 4.5 we conclude that $\psi \left(\frac{\langle v, v \rangle_{g \otimes 1}}{2} \right) = 1$ for all $g \in U_0''$. Hence for all $\alpha \in \mathbb{F}_q$,

$$\psi \left(\frac{\langle v, v \rangle_{\alpha \cdot g \otimes 1}}{2} \right) = \psi \left(\alpha \cdot \left(\frac{\langle v, v \rangle_{g \otimes 1}}{2} \right) \right) = 1.$$

Hence we must have $\langle v, v \rangle_{g \otimes 1} = 0$ for all $g \in U_0''$. So we see that v must in fact lie in $(L_1^\perp/L) \otimes E$, say $v = v' \otimes e$, where $e \in E = \mathbb{F}_{q^2}$. Then e^{q-1} is a norm 1 element of \mathbb{F}_{q^2} , i.e. an element of $SO(E)$. Now suppose $f \in W_1^-$. Then $f((1 \otimes e^{q-1})(v' \otimes e)) = f(v' \otimes e)$ since $f \in W_1$. Hence $f(v' \otimes e^q) = f(v)$. On the other hand, $\Lambda f = -f$, hence $f((1 \otimes \sigma)(v' \otimes e)) = -f(v' \otimes e)$, i.e. $f(v' \otimes e^q) = -f(v)$. Hence we arrive at a contradiction. \square

Proposition 4.7. *The representation W_1^- is cuspidal and degenerate.*

Proof. The lemma implies that none of the unipotent radicals U_0, U_1, U_2 has a fixed vector in W_1^- , or in other words W_1^- is cuspidal. On the other hand, $W_1^-|_{U_0}$ cannot have a one-dimensional U_0 -invariant subspace, for otherwise, the commutator U_0' would have to fix some vector. Hence by 4.4, we conclude that the representation is degenerate. \square

Next, we show that in fact the representation of P_1 on W_1^- is irreducible. For this, let us study the restriction of W_1^- to U_1 . We will need the following:

Lemma 4.8. *Let $v, w \in S'$ i.e. v, w indecomposable. Then $\langle v, v \rangle_{g \otimes 1} = \langle w, w \rangle_{g \otimes 1}$ for all $g \in U_1 \iff v, w$ are in the same $O(E)$ -orbit. For an $O(E)$ -orbit \mathcal{O} in S' , let $\phi_{\mathcal{O}} : U_1 \rightarrow \overline{\mathbb{Q}}_l^*$ be the character $g \mapsto \psi\left(\frac{\langle v, v \rangle_{g \otimes 1}}{2}\right)$ where $v \in \mathcal{O}$.*

Proof. Let $v = v_3 \otimes e_1 + v_4 \otimes e_2$, $w = v_3 \otimes f_1 + v_4 \otimes f_2$ where v_3, v_4 are as before. Then since we have $\langle \cdot, \cdot \rangle_{g \otimes 1} = \langle \cdot, \cdot \rangle_g \otimes \langle \cdot, \cdot \rangle_E$ for all $g \in U_1$, we conclude from the hypothesis of the lemma, that we must have $\langle e_i, e_j \rangle_E = \langle f_i, f_j \rangle_E$. Now v indecomposable $\implies e_1, e_2$ linearly independent. Similarly f_1, f_2 also linearly independent. Hence we conclude that there exists $t \in O(E)$ such that $(1 \otimes t)(v) = w$. The converse is obvious. \square

Lemma 4.9. *Let $\phi : U_1 \rightarrow \overline{\mathbb{Q}}_l^*$ be a character of U_1 . Let $f \in W$. Then U_1 acts on f by $\phi \iff$ for all $v \in L' \otimes E$ such that $f(v) \neq 0$ we have $\psi\left(\frac{\langle v, v \rangle_{g \otimes 1}}{2}\right) = \phi(g)$. Hence $\overline{\mathbb{Q}}_l \cdot f = \langle f \rangle \subset W$ is U_1 -invariant \iff for all $v, w \in L' \otimes E$ where f does not vanish, we have $\langle v, v \rangle_{g \otimes 1} = \langle w, w \rangle_{g \otimes 1}$ for all $g \in U_1$.*

Proof. This is immediate from 4.5. \square

Proposition 4.10. *We have the decomposition $W_1^-|_{U_1} = \bigoplus_i \langle \delta_i \rangle$ as U_1 -modules. Let $\hat{\mathcal{O}}_i \subset S'$ be the $O(E)$ -orbit $\mathcal{O}_i \cup \mathcal{O}'_i$. Then $\langle \delta_i \rangle \cong \phi_{\hat{\mathcal{O}}_i}$ are distinct as U_1 -modules.*

Proof. That $\langle \delta_i \rangle$ is U_1 -invariant follows from 4.9 and one direction of 4.8. The second assertion in the proposition follows from the other direction of 4.8. \square

Let M_1 be the stabilizer in $Sp(V)$ of (L, L') , and let $\mathcal{M} \subset Sp(V \otimes E)$ be the stabilizer of $(L \otimes E, L' \otimes E)$. Then M_1 is a Levi subgroup of P_1 , while \mathcal{M} is a Levi subgroup of \mathcal{P} . So we have $P_1 = M_1 U_1$ and $\mathcal{P} = \mathcal{M} \mathcal{U}$. M_1 can be identified with $GL(L)$ or $GL(L')$, and similarly \mathcal{M} can be identified with $GL(L \otimes E)$ or $GL(L' \otimes E)$. We have $M_1 \hookrightarrow \mathcal{M}$. We now state how \mathcal{M} acts on W .

Proposition 4.11 (See [2], 2.7.). *Let $g \in \mathcal{M}$ be thought of as an element of $GL(L' \otimes E)$. Let $f \in W$. Then for $v \in L' \otimes E$ we have*

$$(g \cdot f)(v) = \chi(\det(g))f(g^{-1}v), \quad (4)$$

where χ is the quadratic character of \mathbb{F}_q^* .

If $g \in M_1$, we have $\det(g \otimes 1) = \det(g)^2$, hence

$$(g \cdot f)(v) = f((g^{-1} \otimes 1)v). \quad (5)$$

It is easy to see that M_1 acts simply transitively on $S' \subset L' \otimes E$. So M_1 acts transitively on $S'/O(E)$, which we can identify with $\{1, 2, \dots, d\}$, and the stabilizer of the element 1, i.e. of $\hat{\mathcal{O}}_1 \in S'/O(E)$ in M_1 is a subgroup O isomorphic to $O(E)$. From above, we see that $g \in M_1$ takes the space $\langle \delta_i \rangle$ to the space $\langle \delta_{g \cdot i} \rangle$. We now apply the little groups method (See [4], section 8.2) in our setting to get the following result.

Proposition 4.12. *The restriction $W_1^-|_{P_1}$ is an irreducible representation of P_1 . It is induced from a one-dimensional character of a certain subgroup of P_1 .*

Remark 4.13. This gives another proof of the irreducibility of W_1^- .

Proof. From what we have proven so far, we see that the characters $\phi_{\phi_i}(\cong \langle \delta_i \rangle \text{ as } U_1\text{-modules})$ of U_1 form a single M_1 -orbit. The stabilizer of $\phi = \phi_{\phi_1}$ in M_1 is the subgroup O isomorphic to $O(E)$. Then from 4.11, we see that O acts on $\langle \delta_1 \rangle$ by the sign representation $\epsilon : O \rightarrow \{\pm 1\}$. Hence the subgroup OU_1 is the stabilizer in P_1 of the one-dimensional subspace $\langle \delta_1 \rangle$. The character ϕ' , or the action of OU_1 on this subspace is given by $\phi'(hg) = \epsilon(h)\phi(g)$ for $h \in O, g \in U_1$. Then we have $W_1^-|_{P_1} \cong \text{Ind}_{OU_1}^{P_1}(\phi')$ and that it is irreducible. \square

5 Deligne-Lusztig Theory

Let us recall some results from Deligne-Lusztig theory that are relevant. Let G be a connected reductive group over $\overline{\mathbb{F}}_q$ provided with an \mathbb{F}_q -structure given by a geometric Frobenius morphism $F : G \rightarrow G$. Let (T_0, B_0) be a pair consisting of an F -stable maximal torus and an F -stable Borel subgroup containing it respectively. Let $B_0 = T_0 U_0$. For an F -stable maximal torus T , let \hat{T}^F denote the group of characters of T^F with values in $\overline{\mathbb{Q}}_l$. Let T, T' be two F -stable maximal tori. We define $N(T, T') = \{g \in G | g^{-1} T g = T'\}$. Define $W(T, T') = T \backslash N(T, T') = N(T, T')/T'$. Note that since T, T' are F -stable, $N(T, T')$ will also be F -stable and we will have an induced action of F on $W(T, T')$. Then using Lang's Theorem, we observe that $W(T, T')^F$ can be identified with $T^F \backslash N(T, T')^F$ or with $N(T, T')^F / T'^F$.

For each integer $n > 0$, we have the norm map $N_n : T'^{F^n} \rightarrow T'^F$.

Definition 5.1 ([1], Defn. 5.5.). Let θ, θ' be characters of T^F, T'^F respectively. We say that (T, θ) and (T', θ') are geometrically conjugate if there exists an integer $n > 0$ and $g \in G^{F^n}$ such that ${}^g T' = T$ and ${}^g(\theta' \circ N_n) = \theta \circ N_n$.

Theorem 5.2 ([1], Cor. 6.3.). *Let θ, θ' be characters of T^F, T'^F respectively. If $(T, \theta), (T', \theta')$ are not geometrically conjugate, then the virtual representations $R_{T, \theta}$ and $R_{T', \theta'}$ are disjoint, i.e. have no irreducible components in common.*

Theorem 5.3 ([1], Thm. 6.8.).

$$\langle R_{T, \theta}, R_{T', \theta'} \rangle = |\{w \in W(T, T')^F | {}^w \theta' = \theta\}|. \quad (6)$$

In particular, $\langle R_{T, \theta}, R_{T', \theta'} \rangle = 0$ if $(T, \theta), (T', \theta')$ are not G^F -conjugate.

Remark 5.4. This does not mean that $R_{T, \theta}$ and $R_{T', \theta'}$ are disjoint, since $R_{T, \theta}, R_{T', \theta'}$ are only virtual characters.

Definition 5.5. We say that a character θ of T^F is in general position, or is regular if it is not fixed by any non-trivial element of $W(T, T)^F$.

Corollary 5.6. *If θ is regular then $\pm R_{T, \theta}$ is irreducible.*

Let St_G denote the Steinberg representation of G^F . For an F -stable torus T , let $\epsilon_T = (-1)^s$, where $s =$ the dimension of the split part of T . We let $\epsilon_G = \epsilon_{T_0}$.

Theorem 5.7 ([1], Thm. 7.1.).

$$\dim(R_{T, \theta}) = Q_T(1) = \epsilon_G \epsilon_T \frac{|G^F|}{|U_0^F| |T^F|} = \epsilon_G \epsilon_T \frac{|G^F|}{|St_G(1)| |T^F|} \quad (7)$$

Let us recall the definition of unipotence.

Definition 5.8. Let ρ be an irreducible representation of the group G^F . We say that ρ is *unipotent* if it occurs in some virtual character $R_{T, 1}$ for some F -stable maximal torus T .

Remark 5.9. By 5.2 we see that a unipotent representation cannot occur in $R_{T,\theta}$ if $\theta \neq 1$.

We will make use of the following result to show that θ_{10} is unipotent.

Proposition 5.10 ([1], Cor. 7.6.). *Let ρ be any virtual character of G^F and let $s \in G^F$ be semisimple. Then*

$$\rho(s) = \frac{1}{St_G(s)} \sum_{T \ni s} \sum_{\theta \in \hat{T}^F} \epsilon_{G \in T} \theta(s) \langle \rho, R_{T,\theta} \rangle. \quad (8)$$

In particular, if s is regular semisimple and if T is the unique maximal torus containing it, then

$$\rho(s) = \sum_{\theta \in \hat{T}^F} \theta(s) \langle \rho, R_{T,\theta} \rangle. \quad (9)$$

6 Unipotence

Let us now return to the case where $G = Sp(4)$. We will make use of the following formula for the values of the character η of the Weil representation on a certain subset of $Sp(V \otimes E)$. This formula was obtained by S.Gurevich and R.Hadani as a consequence of their algebro-geometric approach to the Weil representation (See [3]).

Proposition 6.1 (See [3]; [7], Rem. 1.3.). *Let $g \in Sp(V \otimes E)$ be such that $g - 1$ is invertible. Let χ be the quadratic character of \mathbb{F}_q^* . Then*

$$\eta(g) = \chi(\det(g - 1)). \quad (10)$$

We will make use of this formula to compute $\theta_{10}(s)$, where s is any regular element of a certain F -stable maximal torus T . Let κ be a generator of $\mathbb{F}_{q^4}^*$ and let $\zeta = \kappa^{q^2-1}$. Then let T be an F -stable maximal torus in $Sp(4)$ such that $T^F \cong \langle \zeta \rangle$. (See [5], 3.2.) The group T^F is conjugate in $Sp(4, \overline{\mathbb{F}}_q)$ to the subgroup $H \subset Sp(4, \overline{\mathbb{F}}_q)$ generated by the matrix $\begin{pmatrix} \zeta^q & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 \\ 0 & 0 & \zeta^{-1} & 0 \\ 0 & 0 & 0 & \zeta^{-q} \end{pmatrix} \in Sp(4, \overline{\mathbb{F}}_q)$. $N(T, T)^F$ is conjugate to the subgroup of $Sp(4, \overline{\mathbb{F}}_q)$ generated by H and $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. $W(T, T)^F$ is the cyclic group of order 4, and its generator acts on T^F and \hat{T}^F by taking q -th powers. Let $s \in T^F$ correspond to $\gamma \in \langle \zeta \rangle$. Then by the description of T^F above, it follows that the eigenvalues of $s : V \rightarrow V$ are $\{\gamma, \gamma^q, \gamma^{q^2} = \gamma^{-1}, \gamma^{q^3} = \gamma^{-q}\}$.

Proposition 6.2. *Let $s \in T^F$, $s \neq \pm 1$. Then s is regular semisimple and $\theta_{10}(s) = 1$.*

Proof. W_1^- was defined as the ϵ -isotypic component of W , where $\epsilon : O(E) \rightarrow \{\pm 1\}$ is the ‘sign’ character. Hence we see that for $g \in Sp(V)$ we have

$$\theta_{10}(g) = \frac{1}{2q+2} \sum_{t \in O(E)} \epsilon(t^{-1}) \eta(g \otimes t) = \frac{1}{2q+2} \left(\sum_{t \in SO(E)} \eta(g \otimes t) - \sum_{t \in O(E) \setminus SO(E)} \eta(g \otimes t) \right). \quad (11)$$

In view of this formula, 6.2 would immediately follow once we have the following:

Lemma 6.3. *Let s be as in the Proposition 6.2. Then we have*

$$\eta(s \otimes t) = \begin{cases} 1 & \text{if } t \in SO(E) \\ -1 & \text{if } t \in O(E) \setminus SO(E). \end{cases} \quad (12)$$

Proof. For s as above, and $t \in SO(E)$ (having eigenvalues $\{t, t^q = t^{-1}\}$), $s \otimes t - 1$ is invertible (since $\gamma \neq \pm 1$) with eigenvalues, $\{t\gamma - 1, t\gamma^q - 1, t\gamma^{-1} - 1, t\gamma^{-q} - 1, t^{-1}\gamma - 1, t^{-1}\gamma^q - 1, t^{-1}\gamma^{-1} - 1, t^{-1}\gamma^{-q} - 1\}$.

$$\det(s \otimes t - 1) = \frac{(t\gamma - 1)^2}{t\gamma} \frac{(t\gamma^q - 1)^2}{t\gamma^q} \frac{(t\gamma^{-1} - 1)^2}{t\gamma^{-1}} \frac{(t\gamma^{-q} - 1)^2}{t\gamma^{-q}} = \left(\frac{(t\gamma - 1)(t\gamma^q - 1)(t\gamma^{-1} - 1)(t\gamma^{-q} - 1)}{t^2} \right)^2.$$

Now

$$\frac{(t\gamma - 1)(t\gamma^q - 1)(t\gamma^{-1} - 1)(t\gamma^{-q} - 1)}{t^2} = (t - (\gamma + \gamma^{-1}) + t^{-1})(t - (\gamma^q + \gamma^{-q}) + t^{-1}).$$

Now taking q -th powers, the factors get interchanged, i.e. the above is an element of \mathbb{F}_q . Hence $\det(s \otimes t - 1)$ is a square in \mathbb{F}_q^* , and hence using 6.1, we see that $\eta(s \otimes t) = 1$ for all $t \in SO(E)$.

On the other hand, if $t \in O(E) \setminus SO(E)$, the eigenvalues of t are ± 1 . Hence $s \otimes t - 1$ is invertible with eigenvalues $\{\gamma - 1, \gamma^q - 1, \gamma^{-1} - 1, \gamma^{-q} - 1, -\gamma - 1, -\gamma^q - 1, -\gamma^{-1} - 1, -\gamma^{-q} - 1\}$. Hence

$$\det(s \otimes t - 1) = (1 - \gamma^2)(1 - \gamma^{2q})(1 - \gamma^{-2})(1 - \gamma^{-2q}) = ((\gamma - \gamma^{-1})(\gamma^q - \gamma^{-q}))^2.$$

But now $(\gamma - \gamma^{-1})(\gamma^q - \gamma^{-q}) \notin \mathbb{F}_q$ since it is not fixed by the q -th power map for $\gamma \neq \pm 1$, hence $\eta(s \otimes t) = -1$ for all $t \in O(E) \setminus SO(E)$. Hence we have proved the lemma. □

Hence substituting the values $\eta(s \otimes t)$ in (11), we conclude that $\theta_{10}(s) = 1$ and we have 6.2. □

It is now easy to see that θ_{10} is unipotent.

Proposition 6.4. θ_{10} is unipotent. In fact, we have

$$\langle \theta_{10}, R_{T,1} \rangle = 1. \tag{13}$$

Proof. By 5.10 and 6.2 we see that for $s \in T^F, s \neq \pm 1$

$$\theta_{10}(s) = 1 = \sum_{\theta \in \hat{T}^F} \theta(s) \langle \theta_{10}, R_{T,\theta} \rangle. \tag{14}$$

Note that T^F has only two non-regular characters, namely the trivial character and the quadratic character μ . It is clear that θ_{10} is not one of the $\pm R_{T,\theta}$ corresponding to the regular $\theta \in \hat{T}^F$. This is because, for example, the dimension of $R_{T,\theta}$ is $(q^2 - 1)^2$ (using 5.7), which is not equal to that of θ_{10} . Hence the regular θ do not contribute to the sum. By 5.2 we see that θ_{10} cannot occur in both $R_{T,1}$ and $R_{T,\mu}$, and by the equation above, cannot occur only in $R_{T,\mu}$. Hence the equation just reads $\langle \theta_{10}, R_{T,1} \rangle = 1$. □

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